International Journal of Theoretical Physics, Vol. 44, No. 5, May 2005 (© 2005)
DOI: 10.1007/s10773-005-3983-8

Black Hole Entropies of the Thin Film Model and the Membrane Model Without Cutoffs

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Received August 7, 2004; accepted November 15, 2004

Taking into account the effect of the generalized uncertainty principle on the generalized black hole entropy and tacking the thin film brick-wall model, we calculate the entropy of the quantum scalar field in generalized static black hole. The Bekenstein–Hawking entropies of all well-known static black holes are obtained. The entropy of 2-D membrane just at the event horizon of static black hole is also calculated, and the result of the black hole entropy proportional to the event horizon area can be obtained more easily and generally. This discussion shows that black hole entropy is just identified with the entropy of the quantum field on the event horizon. The difference from the original brick-wall model is that the present result is convergent without any cutoff and the little mass approximation is removed. With residue theorem, the integral difficulty in the calculation of black hole entropy is overcome.

KEY WORDS: generalized uncertainty principle; black hole entropy; event horizon; cutoff.

1. INTRODUCTION

The discovery of Hawking radiation (Hawking, 1974, 1975) affirm the thermal characteristics of black hole and prove that the black hole entropy is proportional to its event horizon area (Bekenstein, 1973, 1974). The origin of black hole entropy has many explanations, in which the brick-wall model (‘t Hooft, 1985) is one of the most representative models. The brick-wall model argues that the black hole entropy is identified with the statistical mechanical entropy arising from the quantum fields propagating outside the event horizon and gives a statistical method of calculating black hole entropy. This method has obviously improved the understanding about the black hole entropy and has been greatly developed (Gao and Liu, 2000; Ghosh and Mitra, 1994; Jing, 1998, 2003; Kastrup, 1997; Li, 2002a,b; Li and Zhao, 2001a,b; Liu *et al*., 2003; Liu and Zhao, 2001; Mukohyama and Israel, 1998; Sun and Liu, 2004). However, in the calculation of original

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brick-wall model, it is discovered that the quantum state density is divergent near the event horizon and the "ultraviolet cutoff" is introduced unnaturally by hand. Further study reveals that the black hole entropy stems from the "wall contribution" (Mukohyama and Israel, 1998), and the brick-wall model is improved to the thin film brick wall model (Gao and Liu, 2000; Li and Zhao, 2001a,b; Liu and Zhao, 2001) and membrane model (Li and Zhao, 2001b). Here, the thin film model takes only the quantum field inside a thin film near the event horizon into account, and the membrane model attributed the black hole entropy to the quantum field on a membrane which is a 2-D surface near the event horizon. In both models, more thermal characteristics of the black holes are given, especially their relation to the event horizon, also the "infrared cutoff" and little mass approximation appearing in the original brick wall model are overcome, but the "ultraviolet cutoff" cannot be avoided yet.

The quantum theory of gravity can transform the Heisenberg uncertainty principle (HUP) into the generalized uncertainty principle (GUP), to which many efforts have been devoted (Adler *et al.*, 2001; Ahlumalia, 2000; Chang *et al.*, 2002; Garay, 1995; Kempf *et al.*, 1995; Rama, 2001). Considering the GUP can seriously influence the density of the states at the Planck temperature (Chang *et al.*, 2002; Rama, 2001), Li (2002a,b) introduce the GUP to the black hole thermodynamics and the calculation of black hole entropy, in which the divergences of state density and entropy of quantum field near the event horizon in the brick-wall model are removed and an upper bound of black hole entropy is obtained. Here, we take into account the effect of the GUP on the state density and the entropy in the generalized static black hole space-time, in which we adopt the thin film brick-wall model and the membrane model, as well as the brick-wall is located at Planck length from the event horizon and the membrane is just at the event horizon. With residue theorem, the integral difficulty appearing in the calculation of black hole entropy (Li, 2002b) is overcome and the entropies themselves of the quantum field with mass inside the thin film brick-wall and on the membrane are calculated separately, and the result that both entropies are proportional to the event horizon area is obtained. Comparing the two results, we find both entropies are similar and the relation between them should be discussed further. Here, as the membrane is just at the event horizon, the thin film brick-wall is cling to the event horizon, both without any cutoff.

2. THE CALCULATION FORMULA FOR THE GENERAL STATIC BLACK HOLE ENTROPY

Adopting the original brick-wall model, Jing (1998) obtains an expression for the generalized static black hole entropy. In this work, the introduction of "ultraviolet cutoff," "infrared cutoff," and little mass approximation are necessary and does not have satisfactory explanations, and applying the expression to the different static black holes, the calculation process and the results are different. Here, by using the GUP, this work can be improved.

In the quantum system under the Planck length, the quantum gravity must be taken into account, so the HUP should be replaced by the GUP. By the GUP, The general position-momentum uncertainty relation is given by

$$\Delta x \, \Delta p \geq \frac{\hbar}{2}[1 + \lambda\langle \hat{p}^2 \rangle + \gamma \langle \hat{x}^2 \rangle + \cdots], \tag{1}$$

where λ, γ are the constants displaying the gravity effect. Considering the momentum play domination and neglecting the high-order small quantities, we have

$$\Delta x \, \Delta p \frac{\hbar}{2}[1 + \lambda\langle \hat{p}^2 \rangle] = \frac{\hbar}{2}\{1 + \lambda[(\Delta p)^2 + \langle \hat{p} \rangle^2]\}. \tag{2}$$

Equation (2) implies that the uncertainty of position should not be infinitesimal, and its minimal value is given by

$$\Delta x_{\min} = \hbar\sqrt{\lambda}, \tag{3}$$

where λ is of order of the Planck area l_p^2.

Based on the GUP, the quantum-states number in the phase-space $d^3\vec{x}d^3\vec{p}$ is given by Chang *et al.* (2002)

$$dN' = \frac{dN}{(1 + \lambda p^2)^3} = \frac{d^3\vec{x}d^3\vec{p}}{(2\pi\hbar)^3(1 + \lambda p^2)^3}. \tag{4}$$

Here dN is the quantum states number based on the HUP. Obviously, the quantum-states number is suppressed by gravity and the suppression is essential on the strongest possible gravitational fields. Of course, while the gravity effect can be neglected, $\lambda = 0$, then $dN' = dN$. As follows, we discuss Eq. (4) near the event horizon of generalized static black hole.

The geometry of generalized static black hole reads

$$ds^2 = g_{tt}\, dt^2 + g_{rr}\, dr^2 + g_{\theta\theta}\, d\theta^2 + g_{\varphi\varphi}\, d\varphi^2. \tag{5}$$

Substituting Eq. (5) into the Klein–Gordon equation of scalar field with mass μ as follows

$$\frac{1}{\sqrt{-g}}\partial_\mu(g^{\mu\nu}\partial_\nu\phi) - \mu^2 = 0, \tag{6}$$

we have

$$g^{tt}\frac{\partial^2\phi}{\partial t^2} + g^{rr}\frac{\partial^2\phi}{\partial r^2} + g^{\theta\theta}\frac{\partial^2\phi}{\partial\theta^2} + g^{\varphi\varphi}\frac{\partial^2\varphi}{\partial\varphi^2} +$$

$$+\frac{1}{\sqrt{-g}}\frac{\partial}{\partial r}(\sqrt{-g}g^{rr})\frac{\partial\phi}{\partial r}+\frac{1}{\sqrt{-g}}\frac{\partial}{\partial\theta}(\sqrt{g}g^{\theta\theta})\frac{\partial\varphi}{\partial\theta}$$
$$+\frac{1}{\sqrt{g}}\frac{\partial}{\partial\varphi}(\sqrt{g}g^{\varphi\varphi})\frac{\partial\varphi}{\partial\varphi}-\mu^2\phi=0. \tag{7}$$

Using the Wenzel–Kramers–Brillouin (WKB) approximation with the ansatz (Jing, 1998)

$$\phi=\exp[-i\omega t+is(r,\theta,\varphi)], \tag{8}$$

and letting $p_r=\frac{\partial s}{\partial r}$, $p_\theta=\frac{\partial s}{\partial\theta}$, $p_\varphi=\frac{\partial s}{\partial\varphi}$, we can obtain the square module of three momentums

$$p^2=g^{rr}p_r^2+g^{\theta\theta}p_\theta^2+g^{\varphi\varphi}p_\varphi^2=-g^{tt}\omega^2-\mu^2. \tag{9}$$

Substituting Eq. (9) into Eq. (4), we obtain the number of quantum states in the phase-space $d^3\vec{x}d^3\vec{p}$ outside the generalized static black hole

$$dN'=\frac{d^3\vec{x}d^3\vec{p}}{(2\pi\hbar)^3[1-\lambda(-g^{tt}\omega^2-\mu^2)]^3}. \tag{10}$$

Near the event horizon, $g_{tt}\to 0$, $(-g^{tt}\omega^2-\mu^2)\to\infty$, then the divergence of the quantum state density based the HUP in the original brick-wall model will possibly be removed by the QUP, thus the free energy and entropy of the quantum field near the event horizon will possibly be calculated without any cutoff. In fact, this has been proved by Li (2002b).

Let us discuss the quantum field covering the event horizon in the thin film $r_h-r_h+\varepsilon$. Setting ε is a small parameter corresponding to the minimal length due to Eq. (3), we have

$$\int_{r_h}^{r_h+\varepsilon}\sqrt{g_{rr}}\,dr=\int_{r_h}^{r_h+\varepsilon}\frac{dr}{\sqrt{g_{tt}}}\approx\int_{r_h}^{r_h+\varepsilon}\frac{dr}{\sqrt{2\kappa(r-r_h)}}=\sqrt{\frac{2\varepsilon}{\kappa}}=\hbar\sqrt{\lambda}, \tag{11}$$

where κ is the surface gravity at the event horizon, in static black hole space–time, it reads $\kappa=2\pi\beta^{-1}$. Based on Eq. (9) and taking the natural units $\hbar=k_{\rm B}=G=c=1$, the number of quantum states with energy less than ω is given

$$\Gamma(\omega)=\frac{1}{(2\pi)^3}\int\frac{dr\,d\theta d\varphi dp_r dp_\theta dp_\varphi}{[1+\lambda(-g^{tt}\omega^2-\mu^2)]^3}$$
$$=\frac{1}{(2\pi)^3}\int\frac{drd\theta d\varphi}{[1+\lambda(-g^{tt}\omega^2-\mu^2)]^3}\int dp_\theta dp_\varphi$$
$$\times\frac{2}{\sqrt{g^{rr}}}\left(-g^{tt}\omega^2-g^{\varphi\varphi}p_\varphi^2-g^{\theta\theta}p_\theta^2-\mu^2\right)^{1/2}$$

$$= \frac{1}{6\pi^2} \int d\theta\, d\varphi \int_{r_h}^{r_h+\varepsilon} \frac{\sqrt{-g}}{\sqrt{-g_{tt}}} (-g^{tt}\omega^2 - \mu^2)^{3/2} \frac{dr}{[1 + \lambda(-g^{tt}\omega^2 - \mu^2)]^3} \tag{12}$$

where the integral goes over those of p_θ and p_φ for which the argument of the square root is positive and the two radial integral direction are taken into account.

According to the theory of canonical ensemble, the free energy of system can be given by 't Hooft (1985)

$$F = \frac{1}{\beta} \sum_{\omega} \ln(1 - e^{-\beta\omega}). \tag{13}$$

In terms of the semiclassical theory and assuming that the energy ω is continuous, we replace the integral by sum, and substitute Eq. (12) into Eq. (13), then

$$\begin{aligned} F(\beta) &= \frac{1}{\beta} \int d\Gamma(\omega) \ln(1 - e^{-\beta\omega}) = -\int_{\mu\sqrt{-g_{tt}}}^{\infty} \frac{\Gamma(\omega)}{e^{\beta\omega} - 1} d\omega \\ &= -\frac{1}{6\pi^2} \int d\theta\, d\varphi \int_{r_h}^{r_h+\varepsilon} \frac{\sqrt{-g}}{\sqrt{-g_{tt}}}\, dr \int_{\mu\sqrt{-g_{tt}}}^{\infty} \frac{(-g^{tt}\omega^2 - \mu^2)^{3/2} d\omega}{(e^{\beta\omega} - 1)[1 + \lambda(-g^{tt}\omega^2 - \mu^2)]^3} \end{aligned} \tag{14}$$

where the integral is taken only those values for which the square-root exists. The entropy of quantum field in the thin film is given by

$$\begin{aligned} S = \beta^2 \frac{\partial F}{\partial \beta} &= \frac{\beta^2}{6\pi^2} \int d\theta\, d\varphi \int_{r_h}^{r_h+\varepsilon} \frac{\sqrt{-g}}{\sqrt{-g_{tt}}}\, dr \\ &\times \int_{\mu\sqrt{-g_{tt}}}^{\infty} \frac{(-g^{tt}\omega^2 - \mu^2)^{3/2} \omega e^{\beta\omega} d\omega}{(e^{\beta\omega} - 1)^2 [1 + \lambda(-g^{tt}\omega^2 - \mu^2)]^3} \end{aligned} \tag{15}$$

Near the event horizon $g^{tt} \to \infty$, then, without any little mass approximation, the integral about ω is reduced to

$$\begin{aligned} \Lambda(\omega) &= \int_{\mu\sqrt{-g_{tt}}}^{\infty} \frac{(-g^{tt}\omega^2 - \mu^2)^{3/2} \omega e^{\beta\omega} d\omega}{(e^{\beta\omega} - 1)^2 [1 + \lambda(-g^{tt}\omega^2 - \mu^2)]^3} \\ &= \int_0^{\infty} \frac{(-g^{tt})^{3/2} \omega^4 d\omega}{(e^{\beta\omega} - 1)(e^{\beta\omega} + 1)(1 - \lambda g^{tt}\omega^2)^3} = \int_0^{\infty} f(\omega) d\omega. \end{aligned} \tag{16}$$

Considering $f(\omega)$ is a even function, setting $-\lambda g^{tt}\omega^2 = x^2$, we have

$$\Lambda(\omega) = \frac{1}{2} \int_{-\infty}^{+\infty} f(\omega)\, d\omega$$

$$= \frac{1}{2\beta^2\lambda^{3/2}} \int_{-\infty}^{+\infty} \frac{a^2x^4dx}{(e^{ax/2} - e^{-ax/2})^2(1 + x^2)^3} = \frac{1}{2\beta^2\lambda^{3/2}} \mathrm{I}(x). \quad (17)$$

where

$$a = \frac{\beta}{\sqrt{-\lambda g^{tt}}}, \quad (18)$$

near the event horizon, $g^{tt} \to \infty$, $a \to 0$.

Due to the integral difficulty of $\Lambda(\omega)$, Li (2002b) only gives an upper bound of black hole entropy. Here, this work can be improved by residue theorem with setting complex function (Sun and Liu, 2004)

$$f(z) = \frac{az^4}{(e^{az/2} - e^{-az/2})^2(1 + z^2)^3}. \quad (19)$$

where $z = 0$ is a removable pole, while $a \neq 2n\pi$, $n = 1, 2, 3, \ldots$, $z = i$ is a third-order pole (deduct several points would not change the result of the integral), its residue is

$$\mathrm{Res} f(i) = \frac{1}{i} \cdot \frac{a^2}{2^8 \sin^4 \frac{a}{2}} \left[-a^2 \left(1 + \cos^2 \frac{a}{2} \right) + 5a \sin a - 6 \sin^2 \frac{a}{2} \right]; \quad (20)$$

$z = \frac{2k\pi i}{a}$, $(k = 1, 2, 3, \ldots)$ is a series second-order poles, their residues are

$$\mathrm{Res} f\left(\frac{2k\pi i}{a}\right) = \frac{1}{i} \sum_k \left(\frac{a}{2k\pi}\right)^3 \frac{\left(\frac{a}{2k\pi}\right)^2 + 2}{\left[\left(\frac{a}{2k\pi}\right)^2 - 1\right]^4}. \quad (21)$$

While $a \to 0$, the sum of residues for $f(z)$in the up half complex coordinate space is $\mathrm{Res} f = \frac{1}{2\pi i} \times \frac{\pi}{16}$. Then, we have

$$\Lambda(\omega) = \frac{1}{2\beta^2\lambda^{3/2}} \mathrm{I}(x) = \frac{\pi}{32\beta^2\lambda^{3/2}}. \quad (22)$$

Substituting Eq. (22) into Eq. (16), we obtain an calculation formula for the Bekenstin–Hawking entropy of generalized static black hole:

$$S = \frac{1}{192\,\pi\lambda^{3/2}} \int d\theta\, d\varphi \int_{r_h}^{r_h+\varepsilon} \sqrt{g g^{tt}}\, dr. \quad (23)$$

3. THE APPLICATION OF THE ENTROPY CALCULATION FORMULA AND ITS 2-D MEMBRANE MODEL

Applying the Eq. (23) to the all well-known static black hole as follows:

Schwarzschild black hole

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - 2M/r} + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\varphi^2, \tag{24}$$

where M is the mass of the black hole;

R–N black hole

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \frac{dr^2}{1 - 2M/r + Q^2/r^2} + r^2 d\theta^2 + r^2 \sin^2\theta\, d\varphi^2, \tag{25}$$

where M is the mass and Q is the magnetic charge of the black hole;

Garfinkle–Horowitz–Strominger Dilatonblack hole (Garfinkle *et al.*, 1991)

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - 2M/r} + r(r-a)(d\theta^2 + \sin^2\theta\, d\varphi^2), \tag{26}$$

where $a = Q^2 \big/ 2Me^{-2\phi_0}$, in which M is the mass, and ϕ_0 is the asymptotic constant of the dilation field;

Static Gibbons–Maeda Dilaton black hole (Gibbons and Maeda, 1988)

$$ds^2 = -\frac{(r - r_+)(r - r_-)}{R^2} dt^2 + \frac{R^2 dr^2}{(r - r_+)(r - r_-)} + R^2(d\theta^2 + \sin^2\theta\, d\varphi^2), \tag{27}$$

where $r_+ = M \pm \sqrt{M^2 + D^2 - P^2 - Q^2}$, $D = (P^2 - Q^2)/2M$ and $R^2 = r^2 - D^2$. The parameters M, Q, and P represent mass, electric charge and magnetic charge, respectively;

Garfinkle–Horne Dilaton black hole (Ghosh and Mitra, 1994)

$$ds^2 = -\left(1 - \frac{r_+}{r}\right)\left(1 - \frac{r_-}{r}\right)^{(1-a^2)/(1+a^2)} dt^2 + \left(1 - \frac{r_+}{r}\right)^{-1}\left(1 - \frac{r_-}{r}\right)^{(a^2-1)/(1+a^2)} dr^2$$
$$+ r^2\left(1 - \frac{r_-}{r}\right)^{2a^2/(1+a^2)} (d\theta^2 + \sin^2\theta\, d\varphi^2), \tag{28}$$

with dilaton field $e^{2\phi} = (1 - r_-/r)^{2a/(1+a^2)} e^{-2\phi o}$ and Maxwell field $F = (Q/r^2) dt \wedge dr$, where a is a coupling constant, $r = r_+$ is the location of the event horizon.

Obviously, with above well-known static black holes, we all have

$$-g^{tt} = g_{rr}, \quad -g = g_{\theta\theta} g_{\varphi\varphi}. \tag{29}$$

Considering the event horizon area $A_h = \int \sqrt{(g_{\theta\theta} g_{\varphi\varphi})_h}\, d\theta\, d\varphi$, so we obtain the Bekenstein-Hawking entropy for all the well-known static black hole all at once

$$S \approx \frac{1}{192\pi\lambda^{3/2}} \int \sqrt{(g_{\theta\theta} g_{\varphi\varphi})_h}\, d\theta\, d\varphi \int_{r_h}^{r_h+\varepsilon} \sqrt{g_{rr}}\, dr = \frac{A_h}{192\pi\lambda}. \tag{30}$$

We believe that the black hole entropy is the reflection of the event horizon characteristic and is just the entropy of the quantum field at the event horizon. For more clearly illustrating this, we further study Eq. (23) using the membrane model (Li and Zhao, 2001b). By calculating the quantum states in the 2-D membrane near the event horizon, Li and Zhao (2001b) obtain the entropy of a black hole, but due to the divergence of quantum state density based on the HUP, the membrane can not be placed on the event horizon. Following study (Liu *et al.*, 2003) shows that the divergence can be removed by the GUP, and the membrane can be placed on the event horizon. But owing to the integral difficulty, Liu *et al.* (2003) only obtain an upper bound of black hole entropy. Here, we improve these works.

From Eq. (1), setting $dr = 0$, the metric of generalized static black hole space–time can be mapped onto the 3-D equal r hypersurface, and the induced metric is described by

$$dS_m = g_{tt}\,dt^2 + g_{\theta\theta}\,d\theta^2 + g_{\varphi\varphi}\,d\varphi^2. \tag{31}$$

Using WKB approximation with $\phi = \exp[-i\omega t + is(r,\theta,\varphi)]$, setting $p_\theta = \partial s/\partial\theta$, $p_\varphi = \partial s/\partial\varphi$, and Substituting Eq. (31) into Eq. (6), we obtain the space square module of two momentums on the hypersurface

$$p^2 = g^{\theta\theta} p_\theta^2 + g^{\varphi\varphi} p_\varphi^2 = -g^{tt}\omega^2 - \mu^2. \tag{32}$$

Based on the GUP, the number of quantum states in the phase-space $d^2\vec{x}d^2\vec{p}$ of the hypersurface is

$$dn' = \frac{dn}{(1+\lambda p^2)^2} = \frac{d^2\vec{x}\,d^2\vec{p}}{(2\pi\hbar)^2(1+\lambda p^2)^2}. \tag{33}$$

where dn is the quantum states number based on the HUP. As discussed in the Section 2, the divergence of dn at the event horizon can be absorbed by the factor $(1+\lambda p^2)^2$, so we can analytically calculate dn' at the event horizon.

Adopting the natural units, based on Eq. (33), the number of quantum states with energy less than ω in the equal$r$membrane outside the event horizon is given by

$$\begin{aligned}
\Gamma(\omega) &= \frac{1}{(2\pi)^2}\int \frac{d\theta\,d\varphi\,dp_\theta\,dp_\varphi}{[1+\lambda(-g^{tt}\omega^2-\mu^2)]^2}\\
&= \frac{1}{(2\pi)^2}\int \frac{d\theta\,d\varphi}{[1+\lambda(-g^{tt}\omega^2-\mu^2)]^2}\int dp_\theta \frac{2}{\sqrt{g^{\varphi\varphi}}}\left(-g^{tt}\omega^2-\mu^2-g^{\theta\theta}p_\theta^2\right)^{1/2}\\
&= \frac{\pi}{(2\pi)^2}\int \frac{d\theta d\varphi(-g^{tt}\omega^2-\mu^2)}{\sqrt{g^{\theta\theta}g^{\varphi\varphi}}[1+\lambda(-g^{tt}\omega^2-\mu^2)]^2}
\end{aligned}$$

$$= \frac{1}{4\pi} \int \frac{(-g^{tt}\omega^2 - \mu^2)\sqrt{g_{\theta\theta} g_{\varphi\varphi}}}{[1 + \lambda(-g^{tt}\omega^2 - \mu^2)]^2} \, d\theta \, d\varphi \tag{34}$$

Considering all the well-known static black hole metric, g^{tt} are all independent onθ, φ, thus, Eq. (34) can be deuced generally to

$$\Gamma(\omega) = \frac{A(-g^{tt}\omega^2 - \mu^2)}{4\pi[1 + \lambda(-g^{tt}\omega^2 - \mu^2)]^2}. \tag{35}$$

where $A = \int \sqrt{g_{\theta\theta} g_{\varphi\varphi}} \, d\theta \, d\varphi$ is the space area of the hypersurface.

According to the theory of canonical ensemble, and applying the semiclassical method, the free energy of the 2-D system can be given by

$$F(\beta) = \frac{1}{\beta} \int d\Gamma(\omega) \ln(1 - e^{-\beta\omega}) = -\int \frac{\Gamma(\omega)}{e^{\beta\omega} - 1} \, d\omega$$
$$= -\frac{A}{4\pi} \int_0^\infty \frac{(-g^{tt}\omega^2 - \mu^2) \, d\omega}{[1 + \lambda(-g^{tt}\omega^2 - \mu^2)]^2 (e^{\beta\omega} - 1)}. \tag{36}$$

Thus, the entropy of quantum states in the membrane is given by

$$S = \beta^2 \frac{\partial F}{\partial \beta} = \frac{\beta^2 A}{4\pi} \int_0^\infty \frac{(-g^{tt}\omega^2 - \mu^2) e^{\beta\omega} \omega \, d\omega}{(e^{\beta\omega} - 1)^2 [1 + \lambda(-g^{tt}\omega^2 - \mu^2)]^2}$$
$$= \frac{\beta^2 A}{4\pi} \int_0^\infty \frac{(-g^{tt}\omega^2 - \mu^2)\omega \, d\omega}{(e^{\beta\omega} - 1)(1 - e^{\beta\omega})[1 + \lambda(-g^{tt}\omega^2 - \mu^2)]^2}. \tag{37}$$

Considering the membrane near the event horizon, then $g^{tt} \to \infty$, thus μ can be neglected naturally, we obtain

$$S = \frac{\beta^2 A}{4\pi} \int_0^\infty \frac{-g^{tt}\omega^3}{(e^{\beta\omega} + e^{-\beta\omega} - 2)(1 - \lambda g^{tt}\omega^2)^2} \, d\omega. \tag{38}$$

Setting $-\lambda g^{tt}\omega^2 = x^2$, considering the integral function of Eq. (38) is a odd function about ω, we have

$$\Lambda(\omega) = \int_0^\infty \frac{-g^{tt}\omega^3 \, d\omega}{(e^{\beta\omega} + e^{-\beta\omega} - 2)(1 - \lambda g^{tt}\omega^2)^2}$$
$$= \frac{1}{\lambda\beta^2} \int_0^\infty \frac{a^2 x^3 \, dx}{(e^{ax} + e^{-ax} - 2)(1 + x^2)^2}$$
$$= \frac{1}{2\lambda\beta^2} \int_{-\infty}^{+\infty} \frac{a^2 |x^3| \, dx}{(e^{ax/2} - e^{-ax/2})^2 (1 + x^2)^2} = \frac{1}{2\lambda\beta^2} \mathrm{I}(x). \tag{39}$$

where $a = \beta \big/ \sqrt{-\lambda g^{tt}}$.

Applying the residue theorem, setting complex function

$$f(z) = \frac{a^2|z^3|}{(e^{az/2} - e^{-az/2})^2(1+z^2)^2} \tag{40}$$

Where $z = 0$ is a removable pole; while $a \neq 2n\pi, n = 1, 2, 3, ...,$ $z = i$, is a third-order pole, its residue is

$$\text{Res} f(i) = \frac{a^3 \cos\frac{a}{2}}{i2^4 \sin^3\frac{a}{2}}. \tag{41}$$

$z = \frac{2k\pi i}{a}(k = 1, 2, 3, \ldots)$ is a series of second-order poles, their residues are

$$\text{Res} f\left(\frac{2k\pi i}{a}\right) = i\sum_k \left(\frac{a}{2k\pi}\right)^2 \frac{\left(\frac{a}{2k\pi}\right)^2 + 3}{\left[\left(\frac{a}{2k\pi}\right)^2 - 1\right]^3}. \tag{42}$$

While $a \to 0$, $\text{Res} f(i) = \frac{1}{2i}$, $\text{Res} f(\frac{2k\pi i}{a}) = 0$, we have

$$\Lambda(\omega) = \frac{1}{2\lambda\beta^2} \times 2\pi i \times \frac{1}{2i} = \frac{\pi}{\lambda\beta^2}. \tag{43}$$

Substituting Eqs. (39) and (43) into Eq. (38), the entropy of quantum states on the membrane near the event horizon is obtained.

$$S = \frac{\beta^2 A}{4\pi} \times \frac{\pi}{\lambda\beta^2} = \frac{A}{4\lambda}. \tag{44}$$

In the generalized static black hole space-time, Applying Eq. (44) to the 3-D null hypersurface namely the event horizon, then A is just the space area of the event horizon A_h, so, by means of calculating the entropy of the quantum states just at the event horizon, we obtain the quantum entropy of all well-known static black holes.

$$S = \frac{A_h}{4\lambda}. \tag{45}$$

4. CONCLUSION AND DISCUSSION

In this paper, by introducing the GUP to the study of black hole entropy and adopting the thin film model and the membrane model, we obtain a entropy calculation formula Eq. (23) and entropy value Eq. (45) for generalized static black hole, without any cutoff and little mass approximation; applying the Eq. (23) to all well-known static black holes, the Bekenstein–Hawking entropies are obtained and the results and calculation process are all the same as showing with the Eq. (30), as the same time, the Eq. (45) is suitable in all well-known static black holes and the result is also proportional to the event horizon area. This is proved that the quantum gravity can overcome the divergent difficulty appearing in the ‘t Hofft’s brick-mall model (include the original thin film model and the original membrane model) and the combination of the GUP with the thin film model and the membrane model are all successful.

Further, as showing in this paper, the membrane is just at the event horizon and the Bekenstein–Hawking entropy is obtained. This can clearly indicate that the black hole entropy is just the entropy of the quantum states on the event horizon. We hold that the result of the membrane model can more deeply reflect the inherent character of black hole entropy, and the difference of Eq. (30) from Eq. (45) (That is to say, the entropy from the event horizon membrane is 48 times of the entropy from the thin film)is just owing to the calculation of Eq. (30) being away from the event horizon. Based on the holography principle (Susskind, 1995), all the black hole characters should be manifested on its boundary namely the event horizon, so the black hole entropy should be obtained by studying the event horizon itself. When the calculation of the entropy of quantum field is away from the event horizon, in their results, there will possibly be some additional factors beyond the black hole entropy, and at the same time, some information about black hole entropy will possibly be lost along with the moving of Hawking radiation from the event horizon to outside. Besides, as well as the Eq. (30) of entropy from the thin film, the entropy expression Eq. (43) of membrane outside the event horizon is also different from the Eq. (44) of entropy from the membrane just at the event horizon; moreover, by the Eq. (18), when the calculation of the entropy of quantum filed is located outside the event horizon with finite distance or infinite distance, a is a finite quantity, Eqs. (21) and (42) are all divergent, then the renormalization of black hole entropy showing in Eqs. (15) and (38) should be necessary (But, the renormalization or the combination of the GUP with the original brick-wall model is still open to problem). Also, the results of original brick-wall model (Jing, 1998; ‘t Hooft, 1985) also include the item of the quantum correction to the black hole entropy which is logarithmically divergent and the item of the contribution from the vacuum surrounding the system at large distance. We think these all can reveal that the calculation of black hole entropy should be at or near the event horizon. More specific and deep physical meaning about this, for example, the variance mode of black hole entropy along with the quantum field away from the event horizon and more deep thermal characteristics and quantum gravity effect of black hole, especially their relation to the event horizon, should be discussed in the future.

ACKNOWLEDGMENTS

The author thanks Profs Zheng Zhao and Jian-Yang Zhu for their zealous help during the research. The author also thanks Dr Xiang Li and Xue-Feng Sun for their discussion on this research. This work is supported by the National Natural Science Foundation of China under Grant No. 10375008.

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